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REMARKS ON THE ASYMPTOTIC MOTIONS OF MECHANICAL SYSTEMS†

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The problem of the existence of asymptotic motions of mechanical systems in the case when the Maclaurin series of the potential energy begins with a permanently positive quadratic form is investigated using the methods described in [1].

1. FIRST we consider the motion of a mechanical system which is described by Lagrange's equations with an analytic Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial x^\cdot}-\frac{\partial L}{\partial x}=0,\ \ x\in R^n;\ \ L(x,x^\cdot)=T(x,x^\cdot)-\Pi(x) \tag{1.1}$$

where $T=\frac{1}{2}\langle K(x)x^\cdot,\ x^\cdot\rangle$ is the kinetic energy ($K(x)$ is a positive definite matrix and $\langle\ ,\ \rangle$ is a scalar product in R_n) and $\Pi(x)$ is the potential energy. Let us assume that system (1.1) has a position of equilibrium which, without any loss of generality, we consider to be the origin of coordinates, and let $\Pi(0)=0$. A motion $x(t)\not\equiv 0$ is referred to as asymptotic motion if $x(t)\to 0$ when $t\to\infty$. By virtue of time reversibility ($x(-t)$ is also a motion), the instability of the equilibrium in the sense of the Lyapunov definition follows from the fact that an asymptotic motion exists.

The hypothesis has been formulated in [2]: if the function $\Pi(x)$ does not have a minimum at the point $x=0$, then an asymptotic motion exists.

The proof of this hypothesis is a complex problem which has been solved under certain additional conditions. The first results in this area were obtained by Kneser [3] while the most powerful results are due to Kozlov [1]. We will supplement these assertions with Theorems 1 and 2 which are presented below and we will then formulate certain generalizations to non-real systems.

Suppose

$$\Pi(x)=\Pi_2(x)+\Pi_j(x)+\ldots\ (2<j) \tag{1.2}$$

are the expansion of the potential energy in a Maclaurin series, Π_i are homogeneous forms of degree i and Π_j is the first non-trivial form after the quadratic form. Henceforth it is assumed that the quadratic form Π_2 has l $(1\leqslant l\leqslant n)$ zero eigenvalues and $n-l$ positive ones. We note that, if $l=0$, the equilibrium is stable and there are no asymptotic motions. We will denote by P the restriction of the function Π in an l-dimensional plane $\pi=\{x:\ \Pi_2(x)=0\}$.

Theorem 1. System (1.1), (1.2) possesses an asymptotic motion if one of the two following conditions is satisfied:

(a) the function $\Pi(x)$ has no minimum at the point $x=0$ and $P\equiv 0$,

(b) the first non-trivial form P_r in the expansion of the function P can take negative values.

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We note that in case (b), the forms $\Pi_j, \ldots, \Pi_{r-1}$ can take both positive and negative values.

When $r=j$, case (b) is identical with the result obtained in [1].

Proof. Normal coordinates can be introduced in the neighbourhood of the point $x=0$ in which (E is a unit matrix)

$$T=\tfrac{1}{2}\langle (E+B(x))x^{\cdot}, x^{\cdot}\rangle,\quad B(0)=0$$

$$\Pi=\tfrac{1}{2}\langle Dy, y\rangle+\Pi_j(x)+\ldots,\quad D=\operatorname{diag}(\lambda_i),\ \lambda_i>0,\quad i=1,\ldots,n-l \tag{1.3}$$

$$x=(y, z), y\in R^{n-l},\quad z\in R^l \tag{1.4}$$

According to the splitting lemma [4], by means of a linear substitution of the form

$$\bar{y}=y+b(x),\quad b(x)=b_{j-1}(x)+b_j(x)+\ldots,\quad \bar{z}=z \tag{1.5}$$

it is possible to reduce expansion (1.4) to the form

$$\bar{\Pi}=\tfrac{1}{2}\langle D\bar{y}, \bar{y}\rangle+W(\bar{z}),\quad W(\bar{z})=W_k(\bar{z})+\ldots, k>2 \tag{1.6}$$

It is clear that

$$P(z)\equiv\tfrac{1}{2}\langle Dc(z), c(z)\rangle+W(z)$$
$$(c(z)=b(y=0, z)=c_m(z)+\ldots, m>j-1) \tag{1.7}$$

It follows from the assumptions relating to case (a) that the function $W(z)\not\equiv 0$ and that it is non-positive. Let the assumptions of case (b) now be satisfied. If $2m>r$, then $k=r$ and $W_r(z)\equiv P_r(z)$. If $2m\leqslant r$, then $k=2m$ and $W_k(z)\leqslant 0$. Consequently, under the assumptions of Theorem 1, the first non-trivial form $W_r(z)$ in the Maclaurin series of the function $W(z)$ takes negative values. Next, we can use the result in [1], according to which asymptotic motions exist, with their asymptotic expansions in the variables $\bar{y}$, $\bar{z}$ of the form

$$\bar{y}=\sum_{i=0}^{\infty}\frac{y_i(\tau)}{t^{2+\mu(2+i)}},\quad \bar{z}=\sum_{i=0}^{\infty}\frac{z_i(\tau)}{t^{\mu(1+i)}},\quad \tau=\ln(t),\quad \mu=\frac{2}{k-2}$$

where y_i and z_i are certain polynomials of τ. The theorem is proved.

We note that the situation when $P_r\geqslant 0$ remains uninvestigated. It is clear that then r is even. If the form P_r is positive definite and $2(j-1)>r$, the potential energy has a local minimum at the equilibrium and there are no asymptotic motions. If $2(j-1)<r$ and $\operatorname{grad}\Pi_{j|x}\not\equiv 0$ then $c_{j-1}\not\equiv 0$ in expression (1.7). Consequently, $k=2(j-1)$ and $W_k\leqslant 0$.

The following theorem is now proved.

Theorem 2. If $P_r\geqslant 0$, $r>2(j-1)$ and $\operatorname{grad}\Pi_{j|x}\not\equiv 0$, then an asymptotic motion exists.

Corollary 1. Under the assumptions of Theorems 1 and 2, the equilibrium $x=0$ is unstable.

2. We will now consider a more general case when, instead of a real system, we consider a system with a semireal Lagrangian

$$L=\tfrac{1}{2}\langle K(x)x^{\cdot}, x^{\cdot}\rangle+\langle v(x), x^{\cdot}\rangle-\Pi(x) \tag{2.1}$$

where $v(i)$ is an analytic vector field in R^n. Without loss of generality, let us assume that $v(x)=0$. The expansion of $v(x)$ in a Maclaurin series has the form $v(x)=v_m(x)+v_{m+1}(x)+\ldots$, $m\geqslant 1$. The remaining assumptions are the same as in Sec. 1.

The following theorem is proved using a procedure similar to that employed in Sec. 1.

Theorem 3. System (1.1), (1.2) possesses an asymptotic motion if one of the following conditions is satisfied:

(a) $m<[r/2]$ and P_r can take negative values,

(b) $m>j-1$, $r>2(j-1)$ and grad $\Pi_{j|x} \not\equiv 0$.

When $r=j$, case (a) follows from the result in [5].

We note that, if $x(t)$ is the motion of a system with the Lagrangian (2.1), then $x(-t)$ is the motion of a system with the Lagrangian $L^{-}=L(x, -x^{\cdot})$ and vice versa. Since the conditions of Theorem 3 are time reversal invariant the following corollary holds.

Corollary 2. Under the above-mentioned assumptions, the position of equilibrium $x=0$ is unstable.

3. Let s constraints, which are linear with respect to the velocities $\langle a_i(x), x^{\cdot}\rangle=0$, $i=1, \ldots, s<n$, where $a_i(x)$ is an analytic vector field in R^n and $a_i(0)\neq 0$, be additionally imposed on a semireal system. The vectors a_i are assumed to be linearly independent. The motion of such a system is described by Lagrange equations with the factors

$$\frac{d}{dt}\frac{\partial L}{\partial x^{\cdot}}-\frac{\partial L}{\partial x}=\Sigma\lambda_i a_i,\quad \langle a_i(x), x^{\cdot}\rangle=0,\quad i=1,\ldots,s \tag{3.1}$$

We denote by $\hat{P}_r$, the restriction of the form P_r in a subspace orthogonal to all the constraints at zero.

Theorem 4. If $m>[r/2]$ and the form $\hat{P}_r$ can take negative values, then an asymptotic motion of system (3.1) exists and the equilibrium $x=0$ is unstable.

When $v(x)\equiv 0$ and $r=j$, Theorem 4 is identical to the result obtained in [6] and, when $\Pi_2\equiv 0$, it is identical to the analytic case of the result in [7].

In order to prove Theorem 4, it is first necessary to expand the potential energy in the form of (1.6) and then use the well-known technique in [6].

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